Multiplication and Division Learning Progression

Number combinations

Early problem-solving: Modeling the action in the problem

In early grades (K, 1st, 2nd, 3rd), students solve real-world problems by directly modeling the action in the problem. They build up their experiences with the four operations through the modeling process.

For example, in the problem "Karen has 20 cookies that she wants to share equally among 4 friends. How many cookies will each friend get?" students usually count out 20 counters to represent the cookies, then put one in each of 4 piles until they are all used up. Then they count the number in a pile for the answer. Students do a large number and variety of these kinds of problems to develop their number combinations – several every day. Through this problem solving, they learn the meaning of multiplication as repeated addition, and they recognize situation that ask "how many groups" and "how many in a group" as division situations. At the same time as they do these problems, they learn how to represent situations with symbols (e.g., in the problem above, $20 \div 4 = 5$ or $4 \times 5 = 20$.)

Fluency: the ability to use mathematics accurately, efficiently, and flexibly

As they get more experience with many types of problems, they remember some of the easier number combinations, and then develop strategies for obtaining other sums, differences and products. The doubling strategy is useful for early addition combinations like $6 + 7$ ("I know that $6 + 6$ is 12, so $6 + 7$ is one more.") Similarly, strategies involving known multiplication combinations are useful for finding unknown ones ("I know that 5×5 is 25, so 6×5 is one more 5, or 30.")

Memory aids: At the same time that students are doing these kinds of problems, they are reinforcing their knowledge of number combinations by playing games where they use number combinations, including skip counting. A good resource for teaching basic multiplication in ways that complement problem solving is here: http://illuminations.nctm.org/LessonDetail.aspx?ID=U109. This set of 4 lessons includes games that use number lines, arrays made from counters, an interactive multiplication chart, an interactive balance to show equivalent combinations, and a product game. On-line simulations are good for one-on-one teaching situations or group work in a computer lab.

Geometric representations of multiplication (array and area models) are useful for those students who are particularly visual (and they are a strong foundation for representing fraction multiplication at later grades). Area models for multiplication also reinforce what students are learning in the measurement and geometry strands about the concepts and computation of area. A good on-line manipulative for number combinations up to 10x10 is here: http://enlvm.usu.edu/ma/nav/activity.jsp?sid=nlvm&cid=2_1&lid=192

Rate and cost problems are other important ways of contextualizing multiplication and division. (e.g. If one item costs $8, how much do 5 items cost? If you eat 20 peas every day, how many peas will you eat in 7 days?)

By the end of third grade, students should be fluent with number combinations (that is, know the fact families) up through 10×10. If some combinations remain uncertain, students can do

targeted work on those combinations, including being taught certain strategies (e.g., the digits in multiples of 9 add up to 9, such as 8 x 9 = 72; or use the two-hand trick. See "Summary of Patterns: No Need to Memorize..." at http://naturalmath.com/mult/mult13.html)

The use of **factors and multiples** in 4th grade is based on skip counting in earlier grades. (Don't emphasize the technical terms "factor" and "multiple" until students understand the concepts. Make sure they can skip count, for example, by 4's, and then say: "All these numbers are *multiples* of 4." After they know the concept, you can use vocabulary strategies to help them remember the name of the concept.)

Multi-digit multiplication

Starting in 4th grade, students learn to use algorithms for carrying out multidigit multiplication and division. Some algorithms are easier to use than others. The learning progression to fluent use of an algorithm involves several key understandings, which are illustrated here with examples:

1. **Multiplication of single digit numbers by powers of 10 and 100, drawing on place value knowledge.** This ability is important for several reasons, including estimating answers (sometimes called "number sense") and using alternative algorithms like the area model described later.

   40 x 6   300 x 4   400 x 7   20 x 40

   Don't just teach the trick of adding zeros. Have students skip count to see what the product is, doing this with several problems, then have them figure out the trick. When you develop the multi-digit algorithm, point out where they are multiplying by 10's and 100's.

2. **Distributive property.** The distributive property underlies the standard multiplication algorithm, since the digits in various places in one number are multiplied separately by the digits in various places in the other number.

   25 x 7 = (20 x 7) + (5 x 7)

   ```
      25
      x7
      35
    +140
     175
   ```

   The distributive property lends itself to the partial product method of multiplication, much easier for many students.

   ```
     46 = 40 + 6
   × 68 = 60 + 8
   ---------------
          2400 = 60 × 40
           360 = 60 ×  6
           320 =  8 × 40
        +1 1 48 =  8 ×  6
          3128
   ```

x	40	6	
60	2400	360	2760
8	320	48	+368
			3128

The partial product method is shown graphically by the area model below.

3. Array model of multiplication

Use this area model to explain why 46 x 25 = 1150

<u>46 x 25</u> = 800 + 200 + 120 + 30 = <u>1150</u>

	40	6	
20	800	120	920
5	200	30	<u>+230</u>
			1150

See http://nlvm.usu.edu/en/nav/frames_asid_192_g_2_t_1.html for an interactive simulation of "rectangle multiplication" including the lattice method, which is an interesting algorithm that is closer to the partial product/area model method.

4. Where does the standard algorithm come from?

Take the example, 27 x 23. View it at the website above, using the "common" algorithm. Where does the 81 come from? Where does the 540 come from?

```
   27
 x 23
   81
 +540
  621
```

The difficulty for some students with the standard algorithm is that there are too many "little" numbers and they get hard to keep track of. Lattice or partial products alleviates this.

5. Estimating

Estimating answers is a very practical skill. In the example above, 27 x 23, students can estimate by rounding 27 up to 30 and then multiplying by 20, which is 600. Rounding one number up and the other down gets a closer estimate.

6. **End goal: Fluency with multiplication (using any workable algorithm).** Fluency means accuracy with efficiency and flexibility. Fluency can be accomplished in different ways. For many students, the "standard" algorithm can be learned so that it can be used with fluency. For many other students, however, when the concepts underlying the standard algorithm aren't learned, common errors occur.

   Identify the mistakes in these two problems and speculate about what the student was thinking:

```
  25        25
 x73       x43
  75       615
 175      820
 250      8815
```

Multi-digit division

Some of the factors that contribute to students' understanding and proficiency with multidigit division are:

1. **Concepts of division:** Division is the inverse operation of multiplication: division "undoes" multiplication. While multiplication is grouping (3 groups of 8 objects = 24 objects), division asks either "how many in a group?" (24 objects divided equally into 3 groups = 8 objects in a group; called partitive division) or "how many groups?" (24 objects divided into groups of 8 each = 3 groups; called measurement division).

   Examples:
   Partitive division: 40 marbles are divided equally among 8 friends. How many marbles does each friend get? Think about the 8 friends as the 8 groups that the marbles will be divided into. How many will be in each group? The question is asking, how many in each group. (8 groups, how many in a group – partitive division.)

   Measurement division: Sue's mother made 75 cookies. She put the cookies in bags, with 3 cookies in each bag. How many bags could she fill up? The number of cookies in a bag is the number in each group, so the question is asking How many groups? The number of cookies in each bag "measures" the total number of cookies, moving each set of 3 cookies into its own group, like 12 inches in a foot ruler measures the number of feet in 72 inches – 6 groups of 12 inches.)

   Another way to think about division is that it "undoes" multiplication. Fact families show this relationship (5, 8, 40 is a fact family for 5 x 8 = 40, 40 ÷ 5 = 8, 40 ÷ 8 = 5). Area representations also show this relationship. Students could create a rectangular area to represent the number to be divided, then mark it off by groups of the divisor; then count the groups.

2. **Problem solving:** Students develop strategies for solving division problems by being given problems like the ones above (both partitive and measurement problems) and asked to solve them in the best way they can. Solving problems over extended periods of time forces students to become more efficient in the strategies they use, as the numbers get larger and larger.

3. **Strategies:** A common strategy for multi-digit division is a form of repeated subtraction, where each partial product is an easy multiple of the divisor. For example, to divide 3129 by 46, the student can guess at the first partial quotient, say 50. 50 x 46 is relatively easy to compute, since multiplying by 50 is like multiplying by 100 and taking half. And then the continue to guess and remove partial quotients, adding them all up when the remainder is less than the divisor. See the example below:

46 ⟌ 3129		
−2300	50	(5s are easy: take half of 10 × 46)
829		
− 460	10	
369		
− 230	5	(I already did it)
139		
− 92	2	(doubling is easy)
47		
− 46	1	
R 1	68	

4. **Remainders:** Students should solve a variety of problems where there are some “left over” and interpret what the left over amount means in the context of the problem. For example, sometimes having some “left over” means that answer is one greater (e.g. How many busses are needed to take 95 students to the zoo if each bus can carry 30 students?) Other times the remainder can be forgotten (e.g. Carrie has 6 friends. She wants to share 32 cookies equally among her friends. How many cookies does each friend get?) Or the remainder could be the answer itself (Carrie shares 32 cookies equally among 6 friends. How many will be left over for her?) Still other times, the amount left over becomes a fraction (in the previous problem, Carrie could divide the 2 left over cookies among the 6 friends, giving each 2/6 or 1/3 of a cookie.) Wait until students have a concept of fractions before jumping into this interpretation of the remainder.

   A good applet for showing division with remainders in the area model is available from NLVM:
   http://nlvm.usu.edu/en/nav/frames_asid_193_g_2_t_1.html?from=category_g_2_t_1.html

See http://www.inghamisd.org/curriculum_math/imat_documents.html for web resources that can help students solidify their number combinations and fluency with operations.